

# 1: Rounding Numbers

**Question:** What is 14 489 to the nearest 1000?

Misconception	Correct
<p>To obtain the answer, round to the nearest 10, 100 and then 1000; thus:</p> <p>14 489 to the nearest 10 is 14 490,            14 490 to the nearest 100 is 14 500,            14 500 to the nearest 1000 is 15 000.</p> <p>Hence: the misconception leads to the incorrect answer, 15 000.</p>	<p>The answer must be either 14 000 or 15 000, and since</p> $14489 - 14000 = 489,$ <p>whilst</p> $15000 - 14489 = 511,$ <p>clearly 14 489 is nearer to 14 000 than to 15 000.</p> <p>Hence: the correct answer is 14 000.</p>

## Further Explanation

The correct explanation above, that 14 489 is nearer to 14 000 than to 15 000, is clear-cut; but why is the 'misconceived' method wrong?

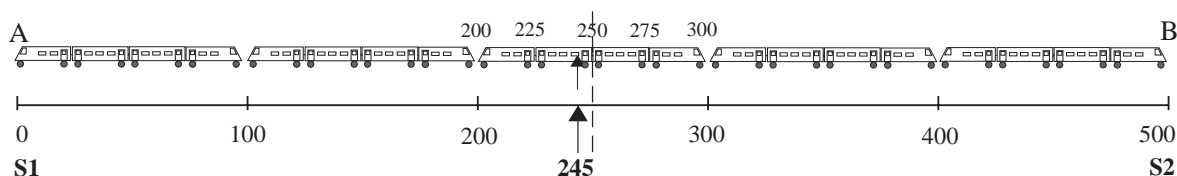
This is best illustrated by taking another example and putting it into context.

Two stations, A and B, are 500 metres apart:



You stand between the stations, 245 m away from A and 255 m from B and you want to walk to the nearest station. Naturally you turn left and walk 245 m.

But now imagine there are five trains parked between the stations, each train 100 m long, and each divided into 4 carriages of equal length.



Your position, 245 m away from A, is now 20 m along the second carriage of the third train, and you decide to find your way to the nearest station by first asking which is the nearest *carriage-end* (the one on the right, 250 metres from A).

Having moved there, you then seek the nearest *end of the train*, which (as, by convention, we round mid-points *up*) is to the right at 300 metres from A. From there finally, you go to the nearest *station*, and this time arrive at B.

Clearly the result is wrong. Why?

Because when you want to find the nearest *station*, the questions about the nearest *carriage-end* and the nearest *train end* are irrelevant. Doing it this way might not matter in some cases, but here it does. The unnecessary interim steps move you away from your starting point, and here, in the wrong direction.

Another instructive point: this exercise provides an opportunity to distinguish between an arbitrary convention (i.e. which way to go from the middle) and a reasoned choice (i.e. the correct approach to round).

### Follow-up Exercises

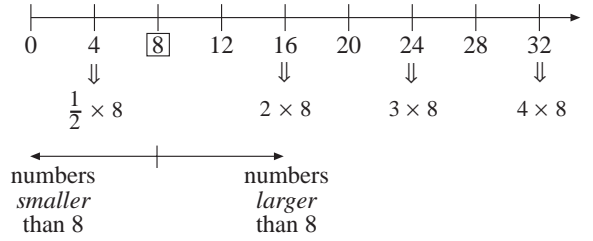
1. Round the following numbers to the nearest thousand:
  - (a) 12 762
  - (b) 9456
  - (c) 493
  - (d) 1488
  
2. Round the following numbers to the nearest hundred:
  - (a) 749
  - (b) 1547
  - (c) 981
  - (d) 2748
  
3. Round the following numbers to the nearest ten thousand:
  - (a) 14 876
  - (b) 24 998
  - (c) 278 376
  - (d) 74 669

### Answers

1. (a) 13 000      (b) 9 000      (c) 0      (d) 1000
2. (a) 700      (b) 1500      (c) 1000      (d) 2700
3. (a) 10 000      (b) 20 000      (c) 280 000      (d) 70 000

## 2: Multiplication can Increase or Decrease a Number

**Question:** Does multiplication always increase a number?

Misconception	Correct
<p>Yes it does; take the number 8, for example:</p> $2 \times 8 = 16$ $3 \times 8 = 24$ $4 \times 8 = 32$ <p>etc.</p> <p>In each it is getting larger, so, yes, multiplication clearly increases a number.</p>	<p>No – it increases a number only under certain conditions.</p> <p>Multiplying any positive number by a whole number greater than 1 will always increase its value – see the example opposite; but consider</p> $\frac{1}{2} \times 8 = 4$ <p>; here the number 8 is reduced.</p> 

### Further Explanation

So, multiplying can have a *reducing* effect when multiplying a positive number by a fraction which is less than one. But this can still be confusing. While we accept the above, the concept of 'a number *times* 8' continues to be perceived as an increase. How then can we attach a meaning to  $\frac{1}{2} \times 8$  so that this will be perceived as decreasing?

When multiplying by a whole positive number, e.g. 6 *times* 5, we understand this as being 5 added over and over again, how ever many *times* – six times in this example. But this interpretation of *times* does not quite work with fractions. If we ask *how many times*, the answer is "*not quite once*".

Again we need to put the term *multiplying* into a context with which we can identify, and which will then make the situation meaningful.

We want to buy 30 roses which are sold in bunches of 5, so we ask for "6 *of* the 5-rose bunches". In this way, the word *times* also often means *of*. If we try using the word *of* when *times* appears to have an unclear meaning, we get  $\frac{1}{2}$  *of* 8 rather than  $\frac{1}{2}$  *times* 8.

Indeed we know what  $\frac{1}{2}$  *of* 8 means – namely 4.

So, by using *of* instead of *times* we are able to understand the concept of multiplying by a fraction and how this can have a reducing effect when the fraction is smaller than 1.

This also helps us to understand how we multiply by a fraction, and why the method works:

the 4 which results from  $\frac{1}{2} \times 8$  (or  $\frac{1}{2}$  *of* 8) can be reached by dividing 8 by 2;

similarly, the 5 which results from  $\frac{1}{3} \times 15$  (or  $\frac{1}{3}$  *of* 15), (or a third *of* fifteen) can be reached by dividing 15 by 3.

Generalising this result gives:

$\frac{1}{d} \times n \text{ is the same as } \frac{n}{d}$
--

## Negative numbers

When your bank balance is +4 pounds you *have* £4.

When your bank balance is -4 pounds you *owe* £4.

*Owing* is the opposite of *having*, so we find that we can associate the concept of 'minus' with '*(the) opposite (of)*'. This also works in reverse.

Thus,  $(-4) \times 8$  means "*owing* £4, eight times over" or "*owing* £32" which is -£32.

Now -32 is smaller than 8, so we have illustrated another case where *multiplying* has a *reducing* effect, i.e. when multiplying by a negative number.

Note that, using the method shown above, it follows that  $-1 \times 8 = -8$ , and vice versa.

### Follow-up Exercises

1. Calculate:

(a)  $1 \times 6$       (b)  $2 \times 6$       (c)  $\frac{1}{2} \times 6$       (d)  $\frac{1}{3} \times 6$

2. Calculate:

(a)  $\frac{1}{4} \times 12$       (b)  $\frac{1}{5} \times 20$       (c)  $\frac{1}{3} \times 18$

3. Are the following statements:

*always true*

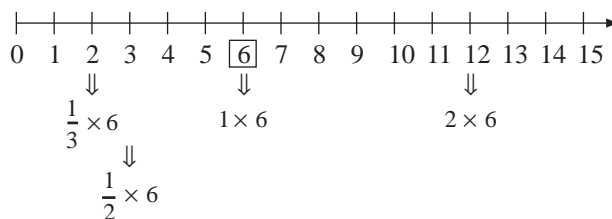
*sometimes false  
and  
sometimes true*

*always false*

- (a) Multiplication of a positive number by a number greater than 1 always increases the number.
- (b) Multiplication of a positive number by a positive number between 0 and 1 always increases the number.
- (c) Multiplication of a negative number by a positive number always increases the first number.

### Answers

1. (a) 6      (b) 12      (c) 3      (d) 2



2. (a) 3      (b) 4      (c) 6

3. (a) Always true
- (b) Always false, as multiplication of a positive number by a number between 0 and 1 will always reduce the number. (e.g.  $\frac{1}{2} \times 12 = 6$ ,  $\frac{1}{3} \times 12 = 4$ , etc.)
- (c) Sometimes false and sometimes true; e.g. for the number  $-8$ ,  $2 \times (-8) = -16$ , so the number is decreased, whereas the number increases in the example below:

$$\frac{1}{2} \times (-8) = -4$$

### 3: Multiplying Decimals

**Question: What is  $0.1 \times 0.1$ ?**

Misconception	Correct
As $1 \times 1 = 1$ then by comparison $0.1 \times 0.1 = 0.1$	The answer is 0.01 as you are multiplying $\frac{1}{10}$ by $\frac{1}{10}$ which means $\frac{1}{10} \times \frac{1}{10}$ . This has value $\frac{1}{100}$ or 0.01 as a decimal.

#### Further Explanation

Consider the simplest case as discussed above

$$0.1 \times 0.1$$

0.1 is the same as  $\frac{1}{10}$  (a tenth), so  $0.1 \times 0.1$  is the same as  $\frac{1}{10}$  *times*  $\frac{1}{10}$ .

But what does this mean?

When we point at boxes containing 6 eggs each and say "3 *of* those boxes please", we walk out with 18 eggs, that is 3 *times* 6.

The meaning of *times* can be therefore be interchanged with *of*.

Hence  $\frac{1}{10}$  *times*  $\frac{1}{10}$  can be seen as *one tenth of one tenth* which is one hundredth or  $\frac{1}{100}$  or 0.01

To summarise:

$$\begin{aligned} 0.1 \times 0.1 &= \frac{1}{10} \text{ times } \frac{1}{10} = \text{one tenth of one tenth} = \text{one hundredth} \\ &= \frac{1}{100} = 0.01 \end{aligned}$$

So

$$0.1 \times 0.1 = 0.01$$

### Misconception 3

#### Generalization

The method that we have used for this simple case can be generalised to multiplying any two decimal numbers together. For example, what is the value of  $54.321 \times 0.06$ ?

We know how to multiply  $54321 \times 6$ , but how do we cope with the decimal points in  $54.321 \times 0.06$  ?

The rule is:

- first perform the multiplication as if there were no decimal points in it, giving here the number 325 926
- then count how many digits are behind (to the right of) the decimal point in both numbers, in this case 5 (3 + 2)
- insert the decimal point that many digits from the right to give the correct answer, in this case, move 5 places from the right to give 3.25926

To understand why the rule works we look at its individual component parts – it is best to tackle this by applying the principal of finding something similar which we can cope with, and then working out the difference between this and the case in hand.

Applying this method, we multiply first the numbers ignoring the decimal point, then examine the effect of having the decimal points as originally given.

(Remember that a number without a decimal point is the same as that number with a decimal point to its right, e.g.  $139 = 139.0$ )

Moving a decimal point one place to the left amounts to *dividing by 10*, e.g. 13.9 is the same as  $\frac{139}{10}$ . (Similarly, moving the point to the right is the same as *multiplying by 10*.)

Moving it again to the left means dividing by 10 again. So, having moved the point 2 places to the left amounts to dividing by 100. Moving it 3 places to the left amounts to dividing by 1000, and so on.

**Note:** What happens if one needs to move the point more positions than there are digits?

For example, if we have the result 76 from the first stage of multiplying 0.19 by 0.04; we now need to put the decimal point 4 positions from the right of 76? To do this we simply add zeros on the left of the number, as many as are needed. So for moving 4 positions to the left in 76, we first write it as 00076 and then move along the decimal point 4 positions from the left to get the answer 0.0076.

Returning to our example of multiplying 54.321 by 0.06, we used 54 321 instead of 54.321, which meant that the number that was 1000 times bigger than the given number, which consequently made the answer 1000 times too big. Further, we continued by using 6 instead of 0.06, a number 100 times bigger than was given, thus making the result yet another 100 times bigger. To bring the result back to what it should have been, we must divide the number 325 926 by 1000 and again by 100 – or doing it in one go, dividing by 100 000 ( $1000 \times 100$ ).

Referring back to our rule, dividing by 100 000 (5 zeros) is the same as moving the point 5 places from the right (remember this 5 came from the 3 digits to the right of the decimal point in 54.321 which were initially ignored and the 2 digits to the right of the decimal point in 0.06).

### Follow-up Exercises

1. Calculate

(a)  $0.1 \times 0.2$

(b)  $0.2 \times 0.2$

(c)  $0.2 \times 0.3$

(d)  $0.5 \times 0.5$

(e)  $0.8 \times 0.2$

(f)  $0.6 \times 0.9$

2. Calculate

(a)  $0.01 \times 0.1$

(b)  $0.02 \times 0.3$

(c)  $0.04 \times 0.6$

(d)  $0.8 \times 0.05$

(e)  $0.01 \times 0.01$

(f)  $0.05 \times 0.07$

3. Calculate

(a)  $1.2 \times 2.3$

(b)  $8.35 \times 1.2$

(c)  $24.7 \times 0.4$

(d)  $1.5 \times 1.5$

(e)  $3.45 \times 2.7$

(f)  $54.3 \times 0.04$

### Answers

1. (a) 0.02      (b) 0.04      (c) 0.06

(d) 0.25      (e) 0.16      (f) 0.54

2. (a) 0.001      (b) 0.006      (c) 0.024

(d) 0.04      (e) 0.0001      (f) 0.0035

3. (a) 2.76      (b) 10.02      (c) 9.88

(d) 2.25      (e) 9.315      (f) 2.172



## 4: Decimals and their Equivalent Fractions

**Question:** What is the equivalent fraction for the decimal 4.422

Misconception	Correct
<p>The common misconception here is that decimals and fractions are different types of numbers.</p> <p>Hence there is no equivalent fraction for this or any other decimal.</p>	<p>You can express the decimal as 4 and (the fraction) <math>\frac{422}{1000}</math> or as <math>\frac{4422}{1000}</math>.</p> <p>These are fraction equivalents.</p> <p>These can be simplified by dividing both numerator and denominator by 2 to give</p> $4.422 = \frac{2211}{500} \text{ or } 4\frac{211}{500}$

### Further Explanation

The method above shows how to obtain an equivalent fraction from a decimal. It is even easier to find the decimal equivalent of a fraction. All you have to do is to use division.

So, for example, the decimal equivalent of the fraction  $\frac{1}{4}$  is 0.25 as 1.00 divided by 4 will give

this result. Similarly the fraction  $\frac{3}{5}$  can be expressed as a decimal by dividing 3 by 5 to obtain 0.6

A similar way of achieving this is to find an equivalent fraction that has either 10 or 100 or 1000 etc. as the denominator. In this way we can determine the fraction equivalent from the resulting numerator. For example,

$$\frac{3}{5} = \frac{3 \times 2}{5 \times 2} = \frac{6}{10}$$

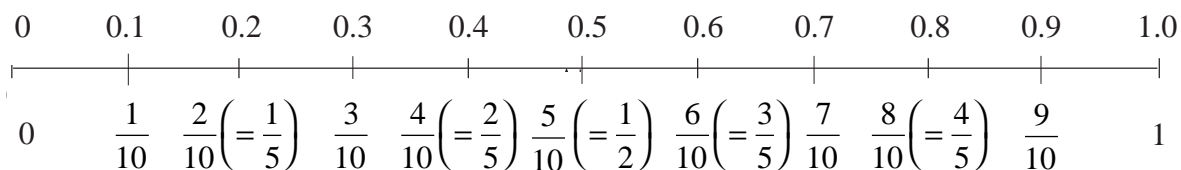
and so the decimal equivalent is 0.6.

Similarly for more complicated fractions such as  $\frac{3}{40}$ . We can find its decimal equivalent by multiplying both numerator and denominator by 25. This gives

$$\frac{3}{40} = \frac{3 \times 25}{40 \times 25} = \frac{75}{1000}$$

and so the decimal equivalent is 0.075

If you are still feeling confused about equivalent decimals and fractions, then think of a number line in which all numbers are represented. A small part of such a number line is shown below with numbers shown as **decimals** above the line and **fractions** below the line



You can readily see, for example, that  $0.2 = \frac{1}{5}$  and  $0.5 = \frac{1}{2}$ .

**Note:** What happens when the fraction cannot be expressed as an equivalent fraction with a numerator of 10 or 100 or 1000 etc.?

An example of this is the fraction  $\frac{1}{3}$ . You can though divide 1.000000..... by 3

to give 0.33333..... This is called a *recurring* decimal, and is often denoted by putting a dot on top of the number (or numbers) that repeat. So we can write

$$\frac{1}{3} = 0.\dot{3}$$

There are many other fractions that give rise to recurring decimals, often with a group of digits that repeat. For example

$$\frac{1}{7} = 0.14285714285714..... = 0.\dot{1}4\dot{2}8\dot{5}7$$

### Follow-up Exercises

1. What are the equivalent fractions for the decimals given below?

- (a) 0.4      (b) 0.6      (c) 0.12      (d) 0.25      (e) 0.125  
 (f) 1.75      (g) 3.24      (h) 21.25  
 (i) 5.875      (j) 4.55

2. What are the equivalent decimals for the fractions given below?

- (a)  $\frac{3}{4}$       (b)  $\frac{3}{20}$       (c)  $\frac{4}{5}$       (d)  $\frac{7}{25}$       (e)  $\frac{9}{40}$   
 (f)  $\frac{23}{20}$       (g)  $\frac{57}{200}$       (h)  $\frac{23}{500}$       (i)  $\frac{223}{25}$       (j)  $\frac{3}{400}$

3. What are recurring decimals for these fractions:

- (a)  $\frac{2}{3}$       (b)  $\frac{1}{6}$       (c)  $\frac{3}{11}$       (d)  $\frac{5}{9}$       (e)  $\frac{4}{7}$

**Answers**

1. (a)  $\frac{2}{5}$  (b)  $\frac{3}{5}$  (c)  $\frac{3}{25}$  (d)  $\frac{1}{4}$  (e)  $\frac{1}{8}$  (f)  $\frac{7}{4}$  (g)  $\frac{81}{25}$   
(h)  $\frac{85}{4}$  (i)  $\frac{47}{8}$  (j)  $\frac{91}{20}$
2. (a) 0.75 (b) 0.15 (c) 0.8 (d) 0.28 (e) 0.225  
(f) 1.15 (g) 0.285 (h) 0.046 (i) 8.92 (f) 0.0075
3. (a) 0.666..... (b) 0.1666..... (c) 0.272727..... (d) 0.5555.....  
(e) 0.57142857142857.....

## 5: Dividing Whole Numbers by Fractions

**Question:** What is the value of  $3 \div \frac{1}{4}$ ?

Misconception	Correct
<p>The value of <math>3 \div \frac{1}{4}</math> is equivalent to <math>3 \div 4</math> and hence has value <math>\frac{3}{4}</math> or 0.75</p> <p>Similarly <math>5 \div \frac{1}{2}</math> is equivalent to <math>5 \div 2</math> and hence has value 2.5 or <math>2\frac{1}{2}</math></p>	<p>The division <math>3 \div \frac{1}{4}</math> means how many <math>\frac{1}{4}</math> are there in the number 3. Clearly there are 4 quarters in 1 and hence <math>3 \times 4 (=12)</math> in 3.</p> <p>So <math>3 \div \frac{1}{4} = 12</math></p>

### Further Explanation

We must learn NEVER to be influenced by what things look like: the meaning of dividing by 2, dividing by 5, etc. is clear: the concept of dividing by a quarter is, however, less straightforward and requires more thought.

Think of  $3 \div \frac{1}{4}$  as 'the number of  $\frac{1}{4}$ 's that fit into 3'.

There are 4 quarters in 1, so in 3 there are  $3 \times 4$  quarters in 3 as can be seen in the diagram below.

1	1	1				
$\frac{1}{4}$	$\frac{1}{4}$					
$\frac{1}{4}$	$\frac{1}{4}$					

So  $3 \div \frac{1}{4}$  (or  $\frac{3}{\frac{1}{4}}$ ) =  $3 \times 4 = 12$ .

Generally  $n \div \frac{1}{m} = n \times m$  or  $\frac{n}{(\frac{1}{m})} = n \times m$

Hence, for example,  $5 \div \frac{1}{2} = 5 \times 2 = 10$ .

**Misconception 5**

There is another way to approach this task *logically* which we will demonstrate with  $6 \div \frac{3}{5}$ .

Use the problem solving method – 'if you are having difficulties, find something similar which you know you CAN do and work out the difference between this and the problem given'.

The difficult part here is *dividing by a fraction*.

Start with something similar which is straightforward: just divide the 6 by 3  $\left(\frac{6}{3}\right)$ . Now continue by examining the effect of the difference between what we did and what was given (using clearer terminology to refer to division, i.e. divide *between*).

We divided the 6 by 3 instead of by the given  $\frac{3}{5}$  (which is, of course, less than 3).

When a cake is divided between a certain number of people, each gets a certain portion. Dividing it between *fewer people* results in each one receiving a larger portion. How much larger? If it is divided between, say, 5 times fewer people, each portions would become 5 times larger.

We arrived at 2 by dividing the 6 by 3. We *should* have divided by something that is 5 times smaller than the 3, (by  $\frac{3}{5}$ ), so, the result should be 5 times *larger* than the  $\frac{6}{3}$ . Thus we

deduce that our  $6 \div \frac{3}{5}$  must mean  $\frac{6}{3} \times 5 (= 10)$ . Generalising,

$$a \div \frac{b}{c} \left( \text{or } \frac{a}{\frac{b}{c}} \right) = \frac{a}{b} \times c \left( = \frac{a \times c}{b} = a \times \frac{c}{b} \right)$$

Yet another way of determining  $\frac{3}{\frac{1}{4}}$  is to forget about the unclear meaning of *dividing by a fraction* and to do whatever yields a result which doesn't contradict other things that are already established.

Whatever we mean by  $\frac{p}{k}$ , we already know that its result,  $r$ , must be such that  $r \times k$  will be equal to  $p$ . i.e. in  $\frac{p}{k} = r$ ,  $r$  must be such that  $r \times k = p$ , (e.g.  $\frac{187}{11}$  is 17 because  $17 \times 11 = 187$ ).

Following this for  $\frac{3}{\frac{1}{4}}$ , we simply seek a result which gives 3 when multiplied by a  $\frac{1}{4}$ .

The question then becomes: "*what times a quarter is 3?*", or using a familiar rephrasing "*a quarter of what is 3?*" (The answer is of course 12.) In summary

$$\text{to determine the value of } r \text{ in } \frac{3}{\frac{1}{4}} = r, \text{ find which value of } r \text{ satisfies } r \times \frac{1}{4} = 3$$

**Follow-up Exercises**

1. Calculate the value of:

(a)  $4 \div \frac{1}{2}$

(b)  $3 \div \frac{1}{3}$

(c)  $6 \div \frac{1}{4}$

(d)  $10 \div \frac{1}{5}$

(e)  $4 \div \frac{1}{3}$

(f)  $5 \div \frac{1}{4}$

(g)  $20 \div \frac{1}{5}$

(h)  $6 \div \frac{1}{6}$

2. Calculate the value of:

(a)  $4 \div \frac{2}{5}$

(b)  $3 \div \frac{3}{4}$

(c)  $10 \div \frac{2}{3}$

(d)  $4 \div \frac{3}{4}$

(e)  $6 \div \frac{3}{5}$

(f)  $1 \div \frac{2}{5}$

(g)  $7 \div \frac{5}{8}$

(h)  $20 \div \frac{4}{5}$

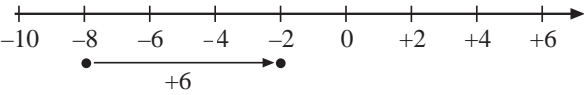
**Answers**

1. (a) 8 (b) 9 (c) 24 (d) 50 (e) 12 (f) 20 (g) 100 (h) 36

2. (a) 10 (b) 4 (c) 15 (d)  $\frac{16}{3}$  (e) 10 (f)  $\frac{5}{2}$  (g)  $\frac{56}{5}$  (h) 25

## 6: Adding with Negative Numbers

**Question:** What is the value of  $-8 + 6$ ?

Misconception	Correct
<p>There are several possible misconceptions including</p> $-8 + 6 = 2$ <p>and</p> $-8 + 6 = -14$	<p>The correct answer is <math>-2</math> as this sum results from adding positive 6 to negative 8, as can be seen from the number line below.</p>  <p><i>Starting</i> at negative 8 and adding 6 you get the answer negative 2.</p> <p>Hence <math>-8 + 6 = -2</math></p>

### Further Explanation

With mathematics, everything is logical; there is no "guessing" and no "maybe", only logical reasoning.

We associate the *minus* sign with *opposite* (as in *give* and *take*).

For example,

$-8$  means *taking away* 8, while  $(+)6$  means *giving* 6

So  $-8 + 6$  translates to

*taking away* 8 and *giving back* 6

which is equivalent to *taking away* only 2

But taking away 2 is what is meant by  $-2$ . That is why  $-8 + 6 = -2$ .

You can argue in a similar way, or do the calculation on a number line, to show that, for example,

$$6 - 8 = -2 \quad \text{and} \quad -6 - 8 = -14$$

Note that the order of giving/taking does not matter.

**Follow-up Exercises**

You might find the number line below helpful when making or checking your calculations.

1. Complete the following:

(a)  $-5 + 3 = \square$

(b)  $-7 + 2 = \square$

(c)  $-9 + 8 = \square$

(d)  $-4 + 4 = \square$

(e)  $-3 + 8 = \square$

(f)  $-5 + 9 = \square$

2. Calculate these values:

(a)  $9 - 7$

(b)  $3 - 5$

(c)  $-4 - 3$

(d)  $5 - 8$

(e)  $-5 - 8$

(f)  $4 - 9$

3. Calculate the value of each of these expressions:

(i)  $a + b$

(ii)  $a - b$

(iii)  $-a + b$

(iv)  $-a - b$

when

(a)  $a = 4$  and  $b = 5$

and

(b)  $a = -4$  and  $b = 3$

**Answers**

1. (a)  $-2$  (b)  $-5$  (c)  $-1$  (d)  $0$  (e)  $5$  (f)  $4$

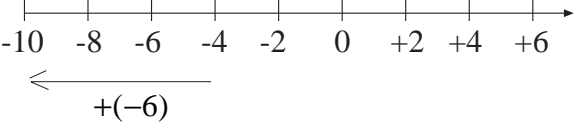
2. (a)  $2$  (b)  $-2$  (c)  $-7$  (d)  $-3$  (e)  $-13$  (f)  $-5$

3. (a)  $9, -1, 1$  and  $-9$  (b)  $-1, -7, 7$  and  $1$



## 7: Calculations with Negative Numbers

**Question:** What is the value that satisfies  $-4 + ? = -10$ ?

Misconception	Correct
There are several possible misconceptions including	The correct answer is $-6$ as this satisfies the sum $-4 + (-6) = -10$ . It can be shown on a number line as given below.
$? = 6$	 <p style="text-align: center;"><math>+(-6)</math></p> <p><i>Starting at negative 4 and adding negative 6 you get the answer negative 10.</i></p> <p>Hence <math>-4 + (-6) = -10</math></p>
and	
$? = 14$	
or even	
$? = -14$	

### Further Explanation

As with the previous misconception (number 6), it is important to realise that there is no guessing, no doubts but straightforward logic. Here we can associate the meaning *owning* when the quantity is positive and *owing* if it negative.

Also, as before, we associate the *minus* sign with *opposite*.

In this example,

I start by owing 4 ( i.e. the ' $-4$ ' )

What is this '+?' that must have happened if I ended up 'owing 10' ?  
( i.e. the ' $-10$ ' )

Another *debt* of 6 must have been added to my plight

We write 'adding a debt of 6' as ' $+(-6)$ '

So in  $-4 + ? = -10$  the ? must be  $-6$

This is equivalent to the use of a number line as shown above; it does not matter which way you argue as long as you get the logic correct!

**Follow-up Exercises**

You might find a number line helpful when making or checking your calculations.

1. Complete the following:

(a)  $-5 - 3 = \square$

(b)  $-7 - 2 = \square$

(c)  $-9 - 8 = \square$

(d)  $-4 - 4 = \square$

(e)  $-3 - 8 = \square$

(f)  $-5 - 9 = \square$

2. Complete these calculations:

(a)  $-5 + \square = -9$

(b)  $-3 + \square = -7$

(c)  $-7 + \square = -10$

(d)  $-8 + \square = -6$

(e)  $-4 + \square = -1$

(f)  $-10 + \square = -6$

3. Calculate the value of each of these expressions:

(i)  $a + b$

(ii)  $a - b$

(iii)  $-a + b$

(iv)  $-a - b$

when

(a)  $a = -3$  and  $b = -7$

and

(b)  $a = 5$  and  $b = -6$

**Answers**

1. (a)  $-8$  (b)  $-9$  (c)  $-17$  (d)  $-8$  (e)  $-11$  (f)  $-14$

2. (a)  $-4$  (b)  $-4$  (c)  $-3$  (d)  $2$  (e)  $3$  (f)  $4$

3. (a)  $-10, 4, -4$  and  $10$  (b)  $-1, 11, -11$  and  $1$

## 8: Calculations with Hundreds and Thousands

**Question:** What is the value of  $2000 \times 300$ ?

Misconception	Correct
There are several possible common incorrect answers including	As
60 000	$2 \times 3 = 6$
or	and
6 000 000	$1000 \times 100 = 100\,000$
	then
	$2000 \times 300 = 600\,000$

### Further Explanation

As before there is only one answer and it is completely logical! We will build on previous knowledge to get the answer; you know, for example, that

$$\begin{aligned} 10 \times 10 &= 100 \\ 10 \times 100 &= 1000 \\ 10 \times 1000 &= 10\,000 \\ 10 \times 10\,000 &= 100\,000 \end{aligned}$$

and so on. Hence it is easy to see that

$$100 \times 1000 = 10 \times \underline{10 \times 1000} = 10 \times \underline{10\,000} = 100\,000$$

and similarly

$$1000 \times 100 = 100\,000$$

We can use this result, but we first note that we can write 2000 and 300 as

$$2000 = 2 \times 1000 \quad \text{and} \quad 300 = 3 \times 100$$

This gives

$$\begin{aligned} 2000 \times 300 &= (2 \times 1000) \times (3 \times 100) \\ &= 2 \times 1000 \times 3 \times 100 \end{aligned}$$

and, as the order of multiplication does not matter, we can rewrite this as

$$\begin{aligned} 2000 \times 300 &= 2 \times 3 \times 1000 \times 100 \\ &= (2 \times 3) \times (1000 \times 100) \\ &= 6 \times 100\,000 \\ &= 600\,000 \end{aligned}$$

**Follow-up Exercises**

1. Calculate:

- (a)  $100 \times 10$       (b)  $100 \times 100$       (c)  $1000 \times 10$   
(d)  $1000 \times 100$       (e)  $1000 \times 1000$       (f)  $100 \times 10000$

2. Calculate:

- (a)  $20 \times 20$       (b)  $20 \times 40$       (c)  $30 \times 40$   
(d)  $50 \times 20$       (e)  $60 \times 60$       (f)  $100 \times 30$

3. Calculate:

- (a)  $200 \times 30$       (b)  $500 \times 20$       (c)  $200 \times 300$   
(d)  $400 \times 300$       (e)  $500 \times 500$       (f)  $1000 \times 40$   
(g)  $2000 \times 30$       (h)  $200 \times 4000$       (i)  $300 \times 4000$   
(j)  $2000 \times 4000$       (k)  $50 \times 40000$       (l)  $3000 \times 7000$

**Answers**

1. (a) 1000      (b) 10000      (c) 10000  
(d) 100000      (e) 1000000      (f) 1000000
2. (a) 400      (b) 800      (c) 1200  
(d) 1000      (e) 3600      (f) 3000
3. (a) 6000      (b) 10000      (c) 60000  
(d) 120000      (e) 250000      (f) 40000  
(g) 60000      (h) 800000      (i) 1200000  
(j) 8000000      (k) 2000000      (l) 21000000

## 9. $0.6 \div 0.05 = ?$

---

As in previous examples, start with something familiar that we can do (by long division), namely

$$\frac{6}{5}, \text{ which is } 1.2$$

Then, examine the effect of the differences between what we did ( $\frac{6}{5}$ ) and what we should have done :

$$\frac{0.6}{0.05}$$

(a) Firstly, we divided a 6, but here we must divide something 10 times smaller (0.6), therefore the resulting answer will be 10 times smaller.

(b) Next, the effect of 5 instead of 0.05: we divided between 5 receivers, but now we must divide between fewer receivers, so the resulting portions will, of course, be greater. How many times greater?

We now use 0.05, which is 100 times smaller than the 5 we used previously: i.e. we now divide the cake between 100 times fewer 'receivers' than previously, so the resulting portions will be 100 times larger.

Taking the effects of (a) and (b) together:

the 1.2 was made 10 times smaller, and then 100 times greater:

net effect: 10 times greater (than 1.2).

So,  $\frac{0.6}{0.05} = 12$ . (We have obtained this result by using logic, not rules).

### **Alternative method:**

The problem arose from dividing by a fraction, 0.05. So, remove this obstacle by replacing the fraction by a manageable 5. This makes the divider 100 times greater. But we must not change the result! So, dividing between 100 times more receivers, we need 100 times more of what we are dividing (the 0.6), that is, we need  $100 \times 0.6$  which is 60.

So, the task of  $\frac{0.6}{0.05}$  becomes  $\frac{60}{5}$ , which, again, is 12.

So, 0.6 cakes divided up between 0.05 people means that they each receive 12 cakes apiece – or do they ?. I wouldn't mind a slice of that !

---

## 10. Are $\frac{1}{2}$ of 6, 0.5 of 6 and 50% of 6 equal or not?

---

0.1 is one tenth. This is because the route from 0 to 1 goes through 0.1, 0.2, 0.3, etc. i.e., 10 steps. Each such step (each 0.1 increment) is therefore 1 tenth (of one).

0.5 is reached after 5 such steps, namely halfway up the 10 steps which led to 1.

This is why  $0.5 = \frac{1}{2}$

Therefore, for instance, 0.5 of 6 =  $\frac{1}{2}$  of 6.

*Note also:*

$\frac{1}{2}$  of 6 is 3 (usually...)

0.5 of 6 is  $0.5 \times 6$  ('times' - 'of' equivalence), and  $0.5 \times 6$  (by long multiplication) is also 3.

### Now, 50% ?

It helps if you consider the percentage sign, %, as a misprint, with the correct version as /100. This is to remind yourself that it means 'divided by 100'.

It is then obvious that 50% is nothing more than  $\frac{50}{100}$

50% is best visualised as dividing 50 (cakes) between 100 (children). Each, then gets one half ( $\frac{1}{2}$ ) (a cake). That is why  $50\% = \frac{1}{2}$  and 50% of 6 is indeed equal to  $\frac{1}{2}$  of 6.

*Note also:* 50% of 6 is  $\frac{50}{100} \times 6$ . Doing the multiplication first:  $\frac{300}{100} (= 3)$ , the same as

$\frac{1}{2}$  of 6.

Now one yard is 36 inches and so,

25% of one yard = 9 inches

and taking square roots of both sides gives,

5% of one yard = 3 inches.

That's right isn't it ? Or would you like it in metric:

1 metre = 100 cm so that,

$$\frac{1}{4} \text{ of one metre} = 25 \text{ cm}$$

and taking square roots of both sides gives,

$$\frac{1}{2} \text{ of one metre} = 5 \text{ cm.}$$

What's wrong ?

## 11: What is $\frac{1}{3}$ of $\frac{1}{3}$ ?

We now know that  $\frac{1}{3}$  of (a)  $\frac{1}{3}$  is  $\frac{1}{3} \times \frac{1}{3}$  ('of' = 'times')

The rule for multiplying fractions is:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

which means that  $\frac{1}{3} \times \frac{1}{3} = \frac{1 \times 1}{3 \times 3} = \frac{1}{9}$ . But where does this rule come from?

Let us apply this rule using the example of the cake:

Think of a piece that is one third ( $\frac{1}{3}$ ) of the cake, i.e. which fits into the whole cake 3 times. Now visualize a slice which is one third of the above piece. Clearly this will fit 9 times into the whole cake, i.e. it is  $\frac{1}{9}$  of the cake. This slice is  $\frac{1}{3}$  of  $\frac{1}{3}$ , the same as  $\frac{1}{3} \times \frac{1}{3}$ .

So we have established why  $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$  which is the same as  $\left(\frac{1 \times 1}{3 \times 3}\right)$ , as in the rule given above.

Had we started with a *fifth* ( $\frac{1}{5}$ ) of the cake, and then taken a slice one third of that fifth, clearly 15 such slices would have fitted into the whole cake, i.e. this slice would be one 15th of the cake (which we can write as  $\frac{1}{3 \times 5}$ , because the 15, of course, came from the 3 x 5).

So we have established that  $\frac{1}{3} \times \frac{1}{5} = \frac{1(\times 1)}{3 \times 5} = \frac{1}{15}$  (again as in the rule).

Now, had we started with  $\frac{1}{5}$  of *seven* cakes (rather than of 1), i.e.  $\frac{7}{5}$ , and then went on, as before, taking one third of the this, then the final piece would have been 7 times bigger (than the above  $\frac{1}{3 \times 5}$ ), namely  $\frac{1(\times 7)}{3 \times 5}$ .



Finally, for the 'sub-division' of the cake, had we taken two thirds ( $\frac{2}{3}$ ), rather than one third, the result

would be yet another 2 times bigger, i.e.  $\frac{2 \times 7}{3 \times 5}$ .

So, we have shown why  $\frac{2}{3} \times \frac{7}{5} = \frac{2 \times 7}{3 \times 5}$ , thus showing where the general rule

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \text{ comes from.}$$

**12:**  $\frac{1}{2} + \frac{1}{4} = \frac{2}{6} ? = \frac{1}{6} ?$       **or what?**

---

The problem is the same as you encounter when doing the sum

1 football player + 1 football team.

How many is it, and how many of *what*?

What we can do is calculate, say,

5 football players + 3 football players    or    2 teams + 4 teams,

namely, add things of the same type. So let us see if we can convert the first sum above to something where we add things of the same *type*. We could do so by replacing the type 'team' with the type 'player'. But this is not just about changing words – we must retain the given contents (that is, the meaning of the word 'team' as a certain number of football-players). Because we know that 1 team means 11 players, we can rewrite the above

1 player + 1 football team      as a manageable task, namely:

1 player + 11 players (12 players)

Note that doing it the other way round, i.e. replacing 'player' by 'team' would be less convenient:

1 player is  $\frac{1}{11}$  of a team, so converting the whole thing into teams would give

$$\frac{1}{11} \text{ team} + 1 \text{ team} = 1\frac{1}{11} \text{ teams (the same as 12 players, but less neat).}$$

If the referee plays for the other team – and he always does - then they are a  $1\frac{1}{11}$  team!

Now we apply the same method to  $\frac{1}{2} + \frac{1}{4}$  one half + one quarter.    Either halves or quarters –

which is more convenient? Should we state how many quarters equal one half or vice versa? The

former, of course: 1 *half* is 2 *quarters*.

So, the  $\frac{1}{2}$  becomes  $2 \times \frac{1}{4}$ , i.e.  $\frac{2}{4}$ , and the above task becomes  $\frac{2}{4} + \frac{1}{4}$ , meaning

$$2 \text{ quarters} + 1 \text{ quarter} = 3 \text{ quarters, i.e. } \frac{3}{4}$$


---

In the above sum, 1 *half* + 1 *quarter*, the different types were associated with the different denominators (dividers), so the problem was solved by a transformation that led to identical denominators. Let us now do this for the more general case:

$\frac{2}{3} + \frac{5}{12}$  (where we have more than one of each type, namely: 2 thirds and 5 twelfths).

Again, we want the same denominators under both, either 3 or 12. Which shall we choose? We must remember that the values of the fractions must remain unchanged, so, if we change the denominator, we must also change the numerator (the divided). If we choose the 3 as the common denominator, we must change the 12 in the second term into a 3, that is, make the denominator 4 times smaller. To keep the value of the fraction unchanged, we must then also make the 5 on top (the numerator) 4 times smaller, resulting in 5/4 as the numerator. This is inconvenient because it creates another fraction above the dividing line.

Instead, we can choose the second denominator, the 12, as the common one. We leave the  $\frac{5}{12}$  as it is and change the first denominator, the 3, to 12. This we do by increasing the 3, by multiplying it by 4. Then, to protect the value of this fraction, we must also make the numerator (the 2) 4 times bigger, this time giving a manageable whole number, 8. *Note that this resulted from choosing the larger of the two denominators as the common one.*

And so, the above  $\frac{2}{3} + \frac{5}{12}$  now becomes  $\frac{8}{12} + \frac{5}{12}$ , 8 twelfths + 5 twelfths = 13 twelfths, i.e.  $\frac{13}{12}$ .

But what about  $\frac{2}{3} + \frac{5}{4}$ ?

To get the first (smaller) denominator (the 3) to be the same as the second denominator 4, the 3 has to be multiplied by 1.33..., and the same must then be done to the 2 above it, which results in a messy 2.66...! So we now need a common denominator which is neither 3 nor 4. In principle, we could choose anything to put equally at the bottom of both fractions. The only problem is that we need to adjust the numerators (to keep the values of the given fractions unchanged) and as we found, we may only do this by multiplying the numerators by whole numbers. From this follows that the denominators, too, can be changed only by multiplying by whole numbers.

The task becomes: By what whole number do we multiply the 3 (the denominator), and by what (different) whole number do we multiply the 4 (the other denominator) so that in both cases we get the same result (namely, the same common denominator)? The nice trick for this is to multiply the first denominator (3) by the second denominator (4) and the second denominator (4) by the first (3) ! (always, of course, multiplying the numerator by the same as the denominator).

So this is what was done:

$$\frac{2 \times 4}{3 \times 4} + \frac{5 \times 3}{4 \times 3}, \text{ which equals } \frac{8}{12} + \frac{15}{12} = \frac{8+15}{12} = \frac{23}{12}$$

Generalizing:

$$\frac{n}{a} + \frac{m}{b} = \frac{n \times b}{a \times b} + \frac{m \times a}{b \times a} = \frac{n \times b + m \times a}{a \times b}$$

A wealthy merchant left his three sons 17 camels – the oldest was to get one half of them, the next son one third and the youngest son to get one ninth. How many did they each get? Left to themselves they'd have been still arguing about it, so wisely they went to an uncle who worked like this. First he lent them a camel so they had 18 camels in all. Then,

$$\frac{1}{2} \times 18 = 9; \quad \frac{1}{3} \times 18 = 6 \quad \text{and} \quad \frac{1}{9} \times 18 = 2;$$

this accounted for  $9+6+2=17$  camels and that meant the uncle could reclaim the one he had lent! But what is  $\frac{1}{2}$ ,  $\frac{1}{3}$  and  $\frac{1}{9}$  of 17?

**13: 900 000 ÷ 300 = ?    30 ÷ 100 = ?**

Write  $900,000 \div 300$  as  $\frac{900000}{300}$ . We are used to seeing the divided – often called the numerator - above the fraction line and the divider (or denominator) below the fraction line. We can write 900 000 as  $9 \times 100\,000$  and 300 as  $3 \times 100$ .

The task then looks like  $\frac{9 \times 100000}{3 \times 100}$ .

Let us first consider  $\frac{9}{3}$ , which equals 3.

We should have divided something 100 000 time bigger than 9, making the result 100 000 times bigger. We should also have divided by something 100 times bigger than 3, making the result 100 times smaller. What results from making something 100 000 times bigger and then 100 times smaller? You can think of the latter as 10 times smaller and then 10 times smaller again, each time knocking one 0 off the 100 000, leaving 1000 as the net 'adjustment' to the 3.

$$\text{So } \frac{900000}{300} = 3000$$

In the same way,  $\frac{9 \times m}{3 \times m}$  is just  $\frac{9}{3}$  because there is the same  $m$ -fold increase as the  $m$ -fold decrease.

Note that this does *not* work with, for example,  $\frac{9+m}{3+m}$  (Try taking  $m = 3$  which gives  $\frac{12}{6}$  which is 2, not 3).

In the case of  $30 \div 100$ , (i.e. 30 %), again we start with  $\frac{3}{1}$  and then make it ten times bigger (because the divided is 30, not 3) and one hundred times smaller (because the divider is 100, not 1). The net adjustment is 10 times smaller than the 3, namely 0.3

Remember: The correct value of a fraction can also always be found by long division.

Try working out as decimals what these are:

$$\frac{1}{7}, \frac{2}{7}, \frac{6}{7}$$

What happens, and why didn't we ask for  $\frac{7}{7}$  ?

## 14: Is $\frac{1}{8} = 0.8$ ? (or 0.08) ?

---

The usual mistake here is to be influenced by the 'looks'. There is absolutely no reason why  $\frac{1}{8}$  should be equal to 0.8. But then again, there's no reason why it shouldn't either.

The fact is it just isn't.

In fact, 0.8 is reached by 8 steps of 0.1 each, and thus means '8 tenths'

$$\text{i.e. } 8 \times \frac{1}{10} \text{ or } \frac{8}{10}.$$

Dividing one cake between 8 people ( $\frac{1}{8}$ ) has little to do with dividing 8 cakes between 10 people ( $\frac{8}{10}$ )

I think I'd rather be one of the 10 people with 8 cakes, than the 8 people with 1 cake.

And by the way, is there any digit  $d$  for which  $\frac{1}{d} = 0.d$  can be true?

**15:  $420 \div 0.7 = 60?$** 

---

The problem here stems from the 0.7. With just a 7 it would be straightforward, so do this first:

$$\frac{420}{7} = 60$$

However, in the division we should have used 0.7 - a number smaller than the 7 we used, in fact, 10 times smaller. So the results should have been 10 times bigger, namely 600.

If you divide by a bigger number the result is smaller, and if you divide by a smaller number the result is bigger.

$$16: \quad \frac{1}{3} \div \frac{1}{9} = \frac{1}{3} ?$$


---

Firstly, whatever the result of dividing  $\frac{1}{3}$  by  $\frac{1}{9}$ , it cannot be  $\frac{1}{3}$  ! The only way you can divide a number and end up where you started, is to divide by one. If, instead of dividing by 1, which would leave the number unchanged, we divide by something 9 times smaller than 1, namely  $\frac{1}{9}$ , the result becomes 9 times bigger. (Dividing between fewer mouths yields larger portions.)

Dividing  $\frac{1}{3}$  by  $\frac{1}{9}$  would make the  $\frac{1}{3}$  9 times bigger,

$$\begin{aligned} \text{i.e. } & \frac{1}{3} \times 9 \\ & = \frac{9}{3} = 3 \end{aligned}$$

$$\text{So, } \frac{1}{3} \div \frac{1}{9} = \frac{1}{3} \times 9 = 3$$

And generally,

$$\frac{k}{\frac{1}{d}} = k \times d$$



17: Is  $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$ ? (because it looks right?)

---

We know that the right hand side,  $\frac{1}{a+b}$ , cannot be equal to the left hand side,  $\frac{1}{a} + \frac{1}{b}$ , because, from Misconception 12, we know that

$$\frac{n}{a} + \frac{m}{b} = \frac{n \times b + m \times a}{a \times b}$$

The  $\frac{1}{a} + \frac{1}{b}$  here is like the example above, with 1s in place of the n and m, so it equals

$$\frac{1 \times b + 1 \times a}{a \times b}$$

that is

$$\frac{b+a}{a \times b},$$

which is nothing like  $\frac{1}{a+b}$ !

We could deduce that  $\frac{1}{a+b}$  cannot equal  $\frac{1}{a} + \frac{1}{b}$  by thinking of what things mean. Start with  $\frac{1}{a}$

Changing this into

$$\frac{1}{a+b}$$

means increasing the divider, thus making the result smaller than  $\frac{1}{a}$ ;

On the other hand,

$$\frac{1}{a} + \frac{1}{b}$$

is bigger than

$$\frac{1}{a},$$

so  $\frac{1}{a+b}$  cannot be equal to  $\frac{1}{a} + \frac{1}{b}$ .

---

## 18: Might 1.25 be greater than 1.4?

---

- maybe, because 125 is bigger than 14?
- or even, because 25 is bigger than 4 ?

Where a digit is positioned is crucially important to how much it contributes to the number.

The further to the left a digit lies, the greater its **weight**.

Consider 2111 and 1999.

Each of these numbers has the same number of digits. 2111 is the larger number because in its leftmost position it has a 2, whereas 1999 has a 1 in that position (and so the contribution of all the 9's to the right of the 1 in 1999 do not help to 'overcome' the 1's in 2111).

The same holds behind the point:

in 1.25 the leftmost position behind the point is 2; 1.4 has the digit 4 in that position, so, (with the same number to the left of the point in both cases), 1.4 is greater than 1.25.

The digit in the leftmost position (behind the point) counts the tenths while the next position to its right counts the hundredths. One tenth (1, being the smallest non-zero digit) is always greater than nine hundredths (the largest digit is 9).

So 1.3 is bigger than 1.29, or even 1.29880789...

**19: Is  $a \times a \times a = 3a$  ? Is  $a^3 = a \times 3$** 

Or, should we *guess what  $a^3$*  is?

In mathematics, each symbol, (e.g.  $a \times 3$ ,  $3a$ ,  $a^3$ ) has a uniquely defined meaning.  $a \times 3$  has been arbitrarily chosen as shorthand for  $a + a + a$ . It cannot mean anything else!

$a^3$  has been, equally arbitrarily, chosen as shorthand for  $a \times a \times a$ . It means precisely this.

$3 \times a$  is shorthand for  $3a$ , but note that the omission of the  $\times$  sign is possible only between a number and a letter (and only in that order). Omitting it from any other combination would lead to something different from a product, e.g.  $33$  is not  $3 \times 3$ ;  $aa$  or  $a3$  means a single variable consisting of two symbols each.

Always consider the (unique!) meaning of the maths you write.

**20:  $2.3 \times 10 = 2.30?$   $20.3?$** 

---

Everything in mathematics is done for a reason – most of the time. But don't expect teachers to obey silly rules like this – they are strictly for the likes of you.

$23 \times 10 = 230$  because 23 means  $20 + 3$ , and the 10 multiplies everything in the 23, i.e. both the 20 and the 3, gives 200 ( $10 \times 20$ ) and 30 ( $10 \times 3$ ) i.e. 230.

It is the same with  $2.3 \times 10$ : 2.3 means  $2 + 0.3$ . Here, too, the 10 multiplies both the 2 and the 0.3, giving 20 ( $10 \times 2$ ) and 3 ( $10 \times 0.3$ ), namely, 23.

So if you want to *estimate* a multiplication there are many ways to do it. Here are some 'guesstimates' for  $28 \times 49$ :

$$28 \times 49 = 28 \times (50 - 1) = 28 \times 50 - 28 \times 1 = 14 \times 100 - 28 = 1400 - 28 = 1372$$

$$28 \times 49 = (30 - 2) \times 49 = 30 \times 49 - 2 \times 49 = 3 \times 490 - 98 = 1372$$

or even

$$28 \times 49 \approx 30 \times 50 \approx 1500 \quad (\approx \text{stands for 'approximately').}$$

Can you easily see whether the accurate answer is larger or smaller than the 1500?

## 21: Definite Way to Deal with Probability

---

*Toss two coins – what is the probability of getting a head and a tail?*

Firstly, beware of intuition. Often, intuition not only fails to give the right answer, it suggests a wrong answer!

*The Method:*

Make a comprehensive list of all the possible outcomes, and count them, (call this sum 't' -for 'total')

Next, count only the outcomes which fit the specification of what you want to find the probability of, (call this count 's' -for 'specified').

The required probability is  $\frac{s}{t}$  (but note : this is true only if each of the possible outcomes is equally likely.)

For the above case, the complete list of the 4 possible outcomes is:

H H  
H T \*  
T H \*  
T T

There are 2 fitting the 'one head, one tail' specification and are marked \*.

Thus the probability is  $\frac{2}{4}$  (=1/2).

*Note:*

If, instead of tossing two coins together, one coin is tossed twice, the probability is the same. (Each possible outcome here refers to a possible combination of outcomes of the 1st and 2nd throw.) However, in this case, if the specification were '1st toss T, 2nd toss H', there is only one such possibility. so its probability is

$$\frac{1}{4}$$

('Probability' of '1' means 'certainty'. One of life's few certainties is that a text on probability would contain an example involving dice...),

So:

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*What is the probability that throwing a pair of dice will yield two 3s?*

As before, you can list all the possible outcome combinations. There are many, so it is easier to think.

One dice can give 6 different numbers. For each of these, the other dice has 6 possibilities. In total,  $6 \times 6$  possible outcomes exist ( $t = 36$ ).

Only one of these is a pair of 3s (or any other equal pair, of course).

So the probability is  $\frac{1}{36}$  (deduced by mathematical reasoning, and not by intuition!).

Finally, we will look at the example of the National Lottery. This is a simple way of illustrating how untrustworthy intuition is.

Would you be likely to fill in the numbers 1, 2, 3, 4, 5, 6 on a lottery line?

Look at it from a ball-bubbling-machine's point of view. Even if it could exercise preference, it doesn't even know what is written on the balls. So, any combination of outcomes is, inevitably, equally (un)likely to occur.

(The only difference that choosing 1, 2, 3, 4, 5, 6 makes, is that, if you win with these numbers, it will make bigger headlines.).

Quite how weird probability can sometimes be is easy to demonstrate. Think about a piece of string, and choosing two points on it and cutting the string there. What is the probability that the pieces of string can make up as the sides of a triangle. It depends.

If you choose one point first, and cut the string; and then randomly choose another point in one of the two pieces of string (and which piece is chosen is itself random), then the probability is

$$\frac{1}{4}$$

However, if you choose your points at the same time and cut the string, the probability is

$$\frac{1}{6}$$

Strange but true.

## 22: How to write $(A + B)^2$ without the $( )$

The explanation consists of two short and simple stages:

(i)  $3 \text{ cats} + 4 \text{ cats} = 7 \text{ cats}.$

The 7 came from (3+4), so we can write:

$$3 \text{ cats} + 4 \text{ cats} = (3 + 4) \text{ cats}$$

We don't need the whole cat, 'C' will do:

$$3 C + 4 C = (3+4) C \quad (3 C \text{ is the same as } 3 \times C)$$

This works just as well with any pair of numbers, say, 'n' and 'p' (instead of the 3 & 4), so:

$$n \times C + p \times C = (n+p) \times C$$

Reversing the order, and using different letters:

$$C \times (a + b) = C \times a + C \times b$$

The 'x' are sometimes omitted or replaced by a dot, i.e.

$$C(a + b) = C \cdot a + C \cdot b$$

Always remember that  $C(a+b)$  is the same as  $(a+b)C$ . This is one of the most common and fundamental tools in Algebra, and now we know where it comes from.

(ii)  $K^2$  is shorthand for  $K \times K$ . This notation works with anything, even with  $\frac{1}{x}$ , namely

$$\frac{1}{x}^2 = \frac{1}{x} \times \frac{1}{x}$$

And what is good for  $\frac{1}{x}$ , is good for  $(A+B)$ :

$$(A + B)^2 = (A + B) \times (A + B).$$

How do we rewrite  $(A + B) \times (A + B)$  without the  $( )$ 's? We know what to do if, in place of the 1st  $(A + B)$  we had a single symbol, like the  $C$  in stage (i), so let us replace, *temporarily*, the 1st  $(A + B)$  with some single symbol,  $\frac{1}{x}$  say:

$$\frac{1}{x} \times (A + B), \text{ which we know is } \frac{1}{x} \times A + \frac{1}{x} \times B$$

Now, return the 1st  $(A + B)$  for the  $\frac{1}{x}$ :

$$(A + B) \times A + (A + B) \times B$$

Here we now have two of these exercises as in (i):  $A \times A + B \times A + A \times B + B \times B$

This can be written as:

$$A^2 + 2AB + B^2$$

So,

$$(A+B)^2 = A^2 + 2AB + B^2 \text{ (all thanks to the } \frac{1}{x} \text{)}$$